A Gauge-invariant Hamiltonian Description of the Motion of Charged Test Particles

Dariusz Chruściński[*]

Institute of Physics, Nicholas Copernicus University

ul. Grudziądzka 5/7, 87-100 Toruń, Poland

and

Jerzy Kijowski[†]

Centrum Fizyki Teoretycznej PAN

Aleja Lotników 32/46, 02-668 Warsaw, Poland

Abstract

New, gauge-independent, second-order Lagrangian for the motion of classical, charged test particles is used to derive the corresponding Hamiltonian formulation. For this purpose a Hamiltonian description of theories derived from the second-order Lagrangian is presented. Unlike in the standard approach, the canonical momenta arising here are explicitly gauge-invariant and have a clear physical interpretation. The reduced symplectic form obtained this way is equivalent to Souriau's form. This approach illustrates a new method of deriving equations of motion from field equation.

1   Introduction

In [1] a new method of deriving equations of motion from field equations was proposed. The method is based on an analysis of the geometric structure of generators of the Poincaré group and may by applied to any special-relativistic, lagrangian field theory. In the case of classical electrodynamics, this method leads uniquely to a manifestly gauge-invariant, second order Lagrangian $\mathcal{L}$ for the motion of charged test particles:

$$\mathcal{L} = L_{particle} + \mathcal{L}_{int} = -\sqrt{1 - \mathbf{v}^2}\, (m - a^\mu u^\nu M_{\mu\nu}^{int}(t, \mathbf{q}, \mathbf{v}))\ , \tag{1}$$

where u^μ denotes the (normalized) four-velocity vector

$$(u^\mu) = (u^0, u^k) := \frac{1}{\sqrt{1 - \mathbf{v}^2}}(1, v^k)\ , \tag{2}$$

---

[*]e-mail: darch@phys.uni.torun.pl

[†]e-mail: kijowski@cft.edu.pl

and $a^\mu := u^\nu \nabla_\nu u^\mu$ is the particle's acceleration (we use the Heaviside-Lorentz system of units with the velocity of light $c = 1$). The skew-symmetric tensor $M_{\mu\nu}^{int}(t, \mathbf{q}, \mathbf{v})$ is equal to the amount of the angular-momentum of the field, which is acquired by our physical system, when the Coulomb field accompanying the particle moving with velocity $\mathbf{v}$ through the space-time point $(t, \mathbf{q})$, is added to the background (external) field. More precisely: the total energy-momentum tensor corresponding to the sum of the background field $f_{\mu\nu}$ and the above Coulomb field decomposes in a natural way into a sum of 1) terms quadratic in the background field, 2) terms quadratic in the Coulomb field 3) mixed terms. The quantity $M_{\mu\nu}^{int}$ is equal to this part of the total angular-momentum $M_{\mu\nu}$, which we obtain integrating only the mixed terms of the energy-momentum tensor.

The above result is a by-product of a consistent theory of interacting particles and fields (cf. [2], [3]), called *Electrodynamics of Moving Particles*.

We have proved in [1] that the new Lagrangian (1) differs from the standard one

$$L = L_{particle} + L_{int} = -\sqrt{1 - \mathbf{v}^2}\,(m - eu^\mu A_\mu(t, \mathbf{q})) \ , \tag{3}$$

by (gauge-dependent) boundary corrections only. Therefore, both Lagrangians generate the same equations of motion for test particles in an external field. In the present paper we explicitly derive these equations and construct the gauge-invariant Hamiltonian description of this theory.

Standard Hamiltonian formalism, based on the gauge-dependent Lagrangian (3), leads to the gauge-dependent Hamiltonian

$$H(t, \mathbf{q}, \mathbf{p}) = \sqrt{m^2 + (\mathbf{p} + e\mathbf{A}(t, \mathbf{q}))^2} + eA_0(t, \mathbf{q}) \ , \tag{4}$$

where the gauge-dependent quantity

$$p_k := p_k^{kin} - eA_k(t, \mathbf{q}) = mu_k - eA_k(t, \mathbf{q}) \tag{5}$$

plays role of the momentum canonically conjugate to the particle's position q^k.

As was observed by Souriau (see [5]), we may replace the above non-physical momentum in the description of the phase space of this theory by the gauge-invariant quantity p^{kin}. The price we pay for this change is, that the canonical contact form, corresponding to the theory of free particles:

$$\Omega = dp_\mu^{kin} \wedge dq^\mu \ , \tag{6}$$

has to be replaced by its deformation:

$$\Omega_S := \Omega - e\,f_{\mu\nu}\,dq^\mu \wedge dq^\nu \ , \tag{7}$$

where e is the particle's charge.

Both Ω and Ω_S are defined on the "mass-shell" of the kinetic momentum, i. e. on the surface $(p^{kin})^2 = -m^2$ in the cotangent bundle T^*M over the space-time M (we use the Minkowskian metric with the signature $(-, +, +, +)$). The forms contain the entire information about dynamics: for free particles the admissible trajectories are those,

whose tangent vectors belong to the degeneracy distribution of Ω. Souriau noticed that replacing (6) by its deformation (7) we obtain the theory of motion of the particle in a given electromagnetic field $f_{\mu\nu}$.

The new approach, proposed in the present paper is based on Lagrangian (1). It leads *directly* to a perfectly gauge-invariant Hamiltonian, having a clear physical interpretation as the sum of two terms: 1) kinetic energy mu_0 and 2) "interaction energy" equal to the ammount of field energy acquired by our physical system, when the particle's Coulomb field is added to the background field.

When formulated in terms of contact geometry, our approach leads uniquely to a new form Ω_N:

$$\Omega_N := \Omega - e\, h_{\mu\nu}\, dq^\mu \wedge dq^\nu \;, \tag{8}$$

where

$$h_{\mu\nu} := 2(f_{\mu\nu} - u_{[\mu}f_{\nu]\lambda}u^\lambda) \tag{9}$$

(brackets denote antisymmetrization), i. e. we prove the following

Theorem 1 *The one dimensional degeneracies of the form Ω_N restricted to the particle's "mass-shell" correspond to the trajectories of a test particle moving in external electromagnetic field.*

It is easy to see that both Ω_S and Ω_N, although different, have the same degeneracy vectors, because h and f give the same value on the velocity vector u_ν:

$$u^\nu h_{\mu\nu} = u^\nu f_{\mu\nu} \;. \tag{10}$$

Hence, both define the same equations of motion. We stress, however, that our Ω_N is *uniquely obtained* from the gauge-invariant Lagrangian (1) *via* the Legendre transformation.

The paper is organized as follows. In section 2 we sketch briefly the (relatively little known) Hamiltonian formulation of theories arising from the second order Lagrangian. In section 3 we prove explicitly that the Euler-Lagrange equations derived from $\mathcal{L}$ are equivalent to the Lorentz equations of motion. Finally, Section 4 contains the gauge-invariant Hamiltonian structure of the theory.

2 Canonical formalism for a 2-nd order Lagrangian theory

Consider a theory described by the 2-nd order lagrangian $L = L(q^i, \dot{q}^i, \ddot{q}^i)$ (to simplify the notation we will skip the index "i" corresponding to different degrees of freedom q^i; extension of this approach to higher order Lagrangians is straightforward). Introducing auxiliary variables $v = \dot{q}$ we can treat our theory as a 1-st order one with lagrangian

constraints $\phi := \dot{q} - v = 0$ on the space of lagrangian variables $(q, \dot{q}, v, \dot{v})$. Dynamics is generated by the following relation:

$$d\, L(q, v, \dot{v}) = \frac{d}{dt}(p\, dq + \pi\, dv) = \dot{p}\, dq + p\, d\dot{q} + \dot{\pi}\, dv + \pi\, d\dot{v} \,. \tag{11}$$

where (p, π) are momenta canonically conjugate to q and v respectively. Because L is defined only on the constraint submanifold, its derivative dL is not uniquely defined and has to be understood as a collection of *all the covectors* which are compatible with the derivative of the function along constraints. This means that the left hand side is defined up to $\mu(\dot{q} - v)$, where μ are Lagrange multipliers corresponding to constraints $\phi = 0$. We conclude that $p = \mu$ is an arbitrary covector and (11) is equivalent to the system of dynamical equations:

$$
\begin{aligned}
\pi &= \frac{\partial L}{\partial \dot{v}} \,, \\
\dot{p} &= \frac{\partial L}{\partial q} \,, \\
\dot{\pi} &= \frac{\partial L}{\partial v} - p \,.
\end{aligned}
\tag{12}
$$

The last equation implies the definition of the canonical momentum p:

$$p = \frac{\partial L}{\partial v} - \dot{\pi} = \frac{\partial L}{\partial v} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right) \,. \tag{13}$$

We conclude, that equation

$$\dot{p} = \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \dot{v}}\right) \,. \tag{14}$$

is equivalent, indeed, to the Euler-Lagrange equation:

$$\frac{\delta L}{\delta q} := \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) + \frac{\partial L}{\partial q} = 0 \,. \tag{15}$$

The hamiltonian description (see e. g. [4]) is obtained from the Legendre transformation applied to (11):

$$-\, dH = \dot{p}\, dq - \dot{q}\, dp + \dot{\pi}\, dv - \dot{v}\, d\pi \,, \tag{16}$$

where $H(q, p, v, \pi) = p\, v + \pi\, \dot{v} - L(q, v, \dot{v})$. In this formula we have to insert $\dot{v} = \dot{v}(q, v, \pi)$, calculated from equation $\pi = \frac{\partial L}{\partial \dot{v}}$. Let us observe that H is linear with respect to the momentum p. This is a characteristic feature of the 2-nd order theory.

In generic situation, Euler-Lagrange equations (15) are of 4-th order. The corresponding 4 hamiltonian equations describe, therefore, the evolution of q and its derivatives up to third order. Due to Hamiltonian equations implied by relation (16), the information about succesive derivatives of q is carried by (v, π, p):

4

- v describes $\dot{q}$

$$\dot{q} = \frac{\partial H}{\partial p} \equiv v \tag{17}$$

hence, the constraint $\phi = 0$ is reproduced due to linearity of H with respect to p,

- π contains information about $\ddot{q}$:

$$\dot{v} = \frac{\partial H}{\partial \pi} \, , \tag{18}$$

- p contains information about $\dddot{q}$

$$\dot{\pi} = -\frac{\partial H}{\partial v} = \frac{\partial L}{\partial v} - p \, , \tag{19}$$

- the true dynamical equation equals

$$\dot{p} = -\frac{\partial H}{\partial q} = \frac{\partial L}{\partial q} \, . \tag{20}$$

3 Equations of motion from the variational principle

In this section we explicitly derive the particle's equations of motion from the variational principle based on the gauge-invariant Lagrangian (1). The Euler-Lagrange equations for a second order Lagrangian theory are given by

$$\dot{p}_k = \frac{\partial \mathcal{L}}{\partial q^k} \, , \tag{21}$$

where, as we have seen in the previous section, the momentum p_k canonically conjugate to the particle's position q^k is defined as:

$$p_k := \frac{\partial \mathcal{L}}{\partial v^k} - \dot{\pi}_k \tag{22}$$

and

$$\pi_k := \frac{\partial \mathcal{L}}{\partial \dot{v}^k} = \frac{1}{\sqrt{1 - \mathbf{v}^2}} \, u^\nu M_{k\nu}^{int}(t, \mathbf{q}, \mathbf{v}) \, . \tag{23}$$

Now,

$$u^\nu M_{k\nu}^{int} = u^0 M_{k0}^{int} + u^l M_{kl}^{int} = -u^0 r_k^{int} + u^l \epsilon_{kl}{}^m s_m^{int} \, , \tag{24}$$

where r_k^{int} and s_m^{int} are the static momentum and the angular momentum of the interaction tensor. They are defined as follows: we consider the sum of the (given) background field $f_{\mu\nu}$ and the boosted Coulomb field $\mathbf{f}_{\mu\nu}^{(y,u)}$ accompanying the particle moving with constant

5

four-velocity u and passing through the space-time point $y = (t, \mathbf{q})$. Being bi-linear in fields, the energy-momentum tensor T^{total} of the total field

$$f^{total}_{\mu\nu} := f_{\mu\nu} + \mathbf{f}^{(y,u)}_{\mu\nu} \tag{25}$$

may be decomposed into three terms: the energy-momentum tensor of the background field T^{field}, the Coulomb energy-momentum tensor $T^{particle}$, which is composed of terms quadratic in $\mathbf{f}^{(y,u)}_{\mu\nu}$ and the "interaction tensor" T^{int}, containing mixed terms:

$$T^{total} = T^{field} + T^{particle} + T^{int} . \tag{26}$$

Interaction quantities (labelled with "int") are those obtained by integrating appropriate components of T^{int}. Because all the three tensors are conserved outside of the sources (i. e. outside of two trajectories: the actual trajectory of our particle and the straight line passing through the space-time point y with four-velocity u), the integration gives the same result when performed over *any* asymptoticaly flat Cauchy 3-surface passing through y.

In particular, r^{int} and s^{int} may be written in terms of the laboratory-frame components of the electric and magnetic fields as follows:

$$r^{int}_k(t, \mathbf{q}, \mathbf{v}) = \int_\Sigma d^3x \ (x_k - q_k)(\mathbf{D}\mathbf{D}_0 + \mathbf{B}\mathbf{B}_0) , \tag{27}$$

$$s^{int}_m(t, \mathbf{q}, \mathbf{v}) = \epsilon_{mij} \int_\Sigma d^3x \ (x^i - q^i)(\mathbf{D} \times \mathbf{B}_0 + \mathbf{D}_0 \times \mathbf{B})^j , \tag{28}$$

where $\mathbf{D}$ and $\mathbf{B}$ are components of the external field f, whereas $\mathbf{D}_0$ and $\mathbf{B}_0$ are components of $\mathbf{f}^{(y,u)}$, i.e.:

$$\mathbf{D}_0(\mathbf{x}; \mathbf{q}, \mathbf{v}) = \frac{e}{4\pi|\mathbf{x} - \mathbf{q}|^3} \frac{1 - \mathbf{v}^2}{\left(1 - \mathbf{v}^2 + \left(\frac{\mathbf{v}(\mathbf{x}-\mathbf{q})}{|\mathbf{x}-\mathbf{q}|}\right)^2\right)^{3/2}} (\mathbf{x} - \mathbf{q}) , \tag{29}$$

$$\mathbf{B}_0(\mathbf{x}; \mathbf{q}, \mathbf{v}) = \mathbf{v} \times \mathbf{D}_0(\mathbf{x}; \mathbf{q}, \mathbf{v}) . \tag{30}$$

It may be easily seen that quantities r^{int}_k and s^{int}_m are not independent. They fulfill the following condition:

$$s^{int}_k = -\epsilon_{kl}{}^m v^l r^{int}_m . \tag{31}$$

To prove this relation let us observe that in the particle's rest-frame (see the Appendix for the definition) the angular momentum corresponding to T^{int} vanishes (cf. [1]). When translated to the language of laboratory frame, this is precisely equivalent to the above relation.

Inserting (31) into (24) we finally get

$$\pi_k = -\left(\delta^l_k + \frac{v^l v_k}{1 - \mathbf{v}^2}\right) r^{int}_l . \tag{32}$$

The quantity r_k^{int} depends upon time *via* the time dependence of the external fields $(\mathbf{D}(t, \mathbf{x}), \mathbf{B}(t, \mathbf{x}))$, the particle's position $\mathbf{q}$ and the particle's velocity $\mathbf{v}$, contained in formulae (29) – (30) for the particle's Coulomb field.

Now, we are ready to compute p_k from (22):

$$
\begin{aligned}
p_k &= \frac{mv_k}{\sqrt{1 - \mathbf{v}^2}} + \dot{v}^l \frac{\partial \pi_l}{\partial v^k} - \left(\frac{\partial \pi_k}{\partial t} + v^l \frac{\partial \pi_k}{\partial q^l} + \dot{v}^l \frac{\partial \pi_k}{\partial v^l} \right) \\
&= p_k^{kin} - \left(\frac{\partial \pi_k}{\partial t} + v^l \frac{\partial \pi_k}{\partial q^l} \right) - \dot{v}^l \left(\frac{\partial \pi_k}{\partial v^l} - \frac{\partial \pi_l}{\partial v^k} \right) \ .
\end{aligned}
\tag{33}
$$

Observe, that the momentum p_k depends upon time, particle's position and velocity but also on particle's acceleration. Hovewer, using (27) one easily shows that due to the

Lemma 1

$$
\frac{\partial \pi_k}{\partial v^l} - \frac{\partial \pi_l}{\partial v^k} = 0 \ ,
\tag{34}
$$

the term proportional to $\dot{v}^l$ vanishes (see Appendix for the proof). Moreover, one can prove the following

Lemma 2

$$
\frac{\partial \pi_k}{\partial t} + v^l \frac{\partial \pi_k}{\partial q^l} = -p_k^{int} \ ,
\tag{35}
$$

where we denote

$$
p_k^{int}(t, \mathbf{q}, \mathbf{v}) = \int_\Sigma d^3x \ (\mathbf{D} \times \mathbf{B}_0 + \mathbf{D}_0 \times \mathbf{B})_k \ .
\tag{36}
$$

For the proof see Appendix. We see that p_k^{int} is the spatial part of the "interaction momentum":

$$
p_\mu^{int}(t, \mathbf{q}, \mathbf{v}) = \int_\Sigma T_{\mu\nu}^{int} \, d\Sigma^\nu \ ,
\tag{37}
$$

where Σ is any hypersurface intersecting the particle's trajectory at the point $(t, \mathbf{q}(t))$. The above integral is well defined (cf. [2]) and it is invariant with respect to changes of Σ, provided the intersection point with the trajectory does not change. It was shown in [1] that p_μ^{int} is orthogonal to the particle's four-velocity, i.e. $p_\mu^{int} u^\mu = 0$.

Finally, the momentum canonically conjugate to the particle's position equals:

$$
p_k = p_k^{kin} + p_k^{int}(t, \mathbf{q}, \mathbf{v}) \ .
\tag{38}
$$

It is a sum of two terms: kinetic momentum p_k^{kin} and the amount of momentum p_k^{int} which is acquired by our system, when the particle's Coulomb field is added to the background (external) field. We stress, that contrary to the standard formulation based on (3), our canonical momentum (38) is gauge-invariant.

Now, Euler-Lagrange equations (21) read

$$\frac{dp_k^{kin}}{dt} + \frac{dp_k^{int}}{dt} = \frac{\partial \mathcal{L}}{\partial q^k} \; , \tag{39}$$

or in a more transparent way:

$$\frac{d}{dt}\left(\frac{mv_k}{\sqrt{1-\mathbf{v}^2}}\right) = -\left(\frac{\partial p_k^{int}}{\partial t} + v^l\frac{\partial p_k^{int}}{\partial q^l}\right) - \dot{v}^l\left(\frac{\partial p_k^{int}}{\partial v^l} - \frac{\partial \pi_l}{\partial q^k}\right) \; . \tag{40}$$

Again, using definitions of π_l and p_k^{int} one shows that due to the following

Lemma 3

$$\frac{\partial p_k^{int}}{\partial v^l} - \frac{\partial \pi_l}{\partial q^k} = 0 \; . \tag{41}$$

the term proportional to the particle's acceleration vanishes (for the proof see Appendix). The last step in our derivation is to calculate $\frac{\partial p_k^{int}}{\partial t} + v^l\frac{\partial p_k^{int}}{\partial q^l}$. In the Appendix we show that the following identities hold:

Lemma 4

$$\frac{\partial p_k^{int}}{\partial t} + v^l\frac{\partial p_k^{int}}{\partial q^l} = -e\sqrt{1-\mathbf{v}^2}\; u^\nu f_{k\nu}(t,\mathbf{q}) = -e(E_k(t,\mathbf{q}) + \epsilon_{klm}v^l B^m(t,\mathbf{q})) \; . \tag{42}$$

Therefore, the term $\frac{\partial p_k^{int}}{\partial t} + v^l\frac{\partial p_k^{int}}{\partial q^l}$ gives exactly the Lorentz force acting on a test particle.

This way we proved that the Euler-Lagrange equations (21) for the variational problem based on $\mathcal{L}$ are equivalent to the Lorentz equations for the motion of charged particles:

$$\frac{d}{dt}\left(\frac{mv_k}{\sqrt{1-\mathbf{v}^2}}\right) = e(E_k(t,\mathbf{q}) + \epsilon_{klm}v^l B^m(t,\mathbf{q})) \; . \tag{43}$$

4 Hamiltonian formulation

By Hamiltonian formulation of the theory we understand, usually, the phase space of Hamiltonian variables $\mathcal{P} = (q,p)$ endowed with the symplectic 2-form $\omega = dp \wedge dq$ and the Hamilton function H (Hamiltonian) defined on $\mathcal{P}$. This function is interpreted as an energy of the system. However, for time-dependent systems this framework is usually replaced by (a slightly more natural) formulation in terms of a *contact form*. For this purpose one considers the *evolution space* $\mathcal{P} \times \mathbf{R}$ endowed with the *contact* 2-form (i.e. closed 2-form of maximal rank):

$$\omega_H := dp \wedge dq - dH \wedge dt \; . \tag{44}$$

In analytical mechanics this form, or rather its "potential" $pdq - Hdt$, is called the *Poincaré-Cartan invariant*. Obviously, ω_H is degenerate on $\mathcal{P} \times \mathbf{R}$ and the one-dimensional

characteristic bundle of ω_H consists of the integral curves of the system in $\mathcal{P} \times \mathbf{R}$. This kind of description may be called the "Heisenberg picture" of classical mechanics. In this picture states are not points in $\mathcal{P}$ but "particle's histories" in $\mathcal{P} \times \mathbf{R}$ (see [5]).

Let us construct the Hamiltonian structure for the theory based on our second order Lagrangian $\mathcal{L}$. Let $\mathcal{P}$ denote the space of Hamiltonian variables, i.e. $(\mathbf{q}, \mathbf{p}, \mathbf{v}, \boldsymbol{\pi})$, where $\mathbf{p}$ and $\boldsymbol{\pi}$ stand for the momenta canonically conjugate to $\mathbf{q}$ and $\mathbf{v}$ respectively. Since our system is manifestly time-dependent (via the time dependence of the external field) we pass to the evolution space endowed with the contact 2-form

$$\Omega_{\mathcal{H}} := dp_k \wedge dq^k + d\pi_k \wedge dv^k - d\mathcal{H} \wedge dt \ , \tag{45}$$

where $\mathcal{H}$ denotes the time-dependent particle's Hamiltonian.

To find $\mathcal{H}$ on $\mathcal{P} \times \mathbf{R}$ one has to perform the (time-dependent) Legendre transformation $(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}, \dot{\mathbf{v}}) \to (\mathbf{q}, \mathbf{p}, \mathbf{v}, \boldsymbol{\pi})$, i.e. one has to calculate $\dot{\mathbf{q}}$ and $\dot{\mathbf{v}}$ in terms of Hamiltonian variables from formulae:

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \dot{\pi}_k \ , \qquad \pi_k = \frac{\partial \mathcal{L}}{\partial \dot{v}^k} \ . \tag{46}$$

This transformation is singular due to linear dependence of $\mathcal{L}$ on $\dot{\mathbf{v}}$ and gives rise to the time-dependent constraints, given by equations (23) and (38). The constraints can be easily solved i.e. momenta p_k and π_k can be uniquely parameterized by the particle's position q^k, velocity v^k and the time t. Let $\mathcal{P}^*$ denote the constrained submanifold of the evolution space $\mathcal{P} \times \mathbf{R}$ parametrized by $(\mathbf{q}, \mathbf{p}^{kin}, t)$. The reduced Hamiltonian on $\mathcal{P}^*$ reads:

$$\mathcal{H}(t, \mathbf{q}, \mathbf{v}) = p_k v^k + \pi_k \dot{v}^k - \mathcal{L} = \frac{m}{\sqrt{1 - \mathbf{v}^2}} + v^k p_k^{int}(t, \mathbf{q}, \mathbf{v}). \tag{47}$$

Due to identity $u^\mu p_\mu^{int} = 0$ (cf. [1]) we have

$$\mathcal{H}(t, \mathbf{q}, \mathbf{v}) = \frac{m}{\sqrt{1 - \mathbf{v}^2}} - p_0^{int}(t, \mathbf{q}, \mathbf{v}) \ , \tag{48}$$

and, therefore,

Theorem 2 *The particle's Hamiltonian equals to the "$-p_0$" component of the following, perfectly gauge-invariant, four-vector*

$$p_\mu := p_\mu^{kin} + p_\mu^{int}(t, \mathbf{q}, \mathbf{v}) = m u_\mu + p_\mu^{int}(t, \mathbf{q}, \mathbf{v}). \tag{49}$$

Using the laboratory-frame components of the external electromagnetic field we get:

$$p_0^{int}(t, \mathbf{q}, \mathbf{v}) = - \int d^3x \ (\mathbf{D} \mathbf{D}_0 + \mathbf{B} \mathbf{B}_0) \ . \tag{50}$$

Now, let us reduce the contact 2-form (45) on $\mathcal{P}^*$. Calculating $p_k = p_k(\mathbf{q}, \mathbf{p}^{kin}, t)$ and $\pi_k = \pi_k(\mathbf{q}, \mathbf{p}^{kin}, t)$ from (23) – (38) and inserting them into (45) one obtains after a simple algebra:

$$\Omega_N = dp_\mu^{kin} \wedge dq^\mu - e \, h_{\mu\nu} \, dq^\mu \wedge dq^\nu \ , \tag{51}$$

where $q^0 \equiv t$ and $h_{\mu\nu}$ is the following 4-dimensional tensor:

$$e\, h_{\mu\nu}(t, \mathbf{q}, \mathbf{v}) := \frac{\partial p_\nu^{int}}{\partial q^\mu} - \frac{\partial p_\mu^{int}}{\partial q^\nu} \ . \tag{52}$$

Using techniques presented in the Appendix one easily proves

Lemma 5

$$\frac{\partial p_\nu^{int}}{\partial q^\mu} = e \Pi_\mu{}^\lambda f_{\lambda\nu} \ , \tag{53}$$

where

$$\Pi_\mu{}^\lambda := \delta_\mu{}^\lambda + u_\mu u^\lambda \tag{54}$$

is the projection on the hyperplane orthogonal to u^μ (i.e. to the particle's rest-frame hyperplane, see the Appendix). Therefore

$$h_{\mu\nu} = \Pi_\mu{}^\lambda f_{\lambda\nu} - \Pi_\nu{}^\lambda f_{\lambda\mu} = 2(f_{\mu\nu} - u_{[\mu}f_{\nu]\lambda}u^\lambda) \ . \tag{55}$$

where $a_{[\alpha}b_{\beta]} := \frac{1}{2}(a_\alpha b_\beta - a_\beta b_\alpha)$. The form Ω_N is defined on a submanifold of cotangent bundle T^*M defined by the particle's "mass shell" $(p^{kin})^2 = -m^2$.

Observe, that the 2-form (51) has the same structure as the Souriau's 2-form (7). They differ by the "curvature" 2-forms f and h only. However, the difference "$h - f$" vanishes identically along the particle's trajectories due to the fact that both $f_{\mu\nu}$ and $h_{\mu\nu}$ have the same projections in the direction of u^μ (see formula (10)). We conclude that the characteristic bundle of Ω_N and Ω_S are the same and they are described by the following equations:

$$\dot{q}^k = v^k \ , \tag{56}$$
$$\dot{v}^k = \sqrt{1 - \mathbf{v}^2}\, \frac{e}{m}(g^{kl} - v^k v^l)(E_l + \epsilon_{lij}v^i B^j) \ , \tag{57}$$

which are equivalent to the Lorentz equations (43).

We have two different contact structures which have the same characteristic bundles. Therefore, from the physical point of view, these forms are completely equivalent.

Appendix

Due to the complicated dependence of the Coulomb field $\mathbf{D}_0$ and $\mathbf{B}_0$ on the particle's position $\mathbf{q}$ and velocity $\mathbf{v}$, formulae containing the respective derivatives of these fields are rather complex. To simplify the proofs, we shall use for calculations the particle's rest-frame, instead of the laboratory frame. The frame associated with a particle moving along a trajectory ζ may be defined as follows (cf. [3], [1]): at each point $(t, \mathbf{q}(t)) \in \zeta$ we take the 3-dimensional hyperplane Σ_t orthogonal to the four-velocity u^μ (the *rest-frame hypersurface*). We parametrize Σ_t by cartesian coordinates (x^k), $k = 1, 2, 3$, centered at

the particle's position (i.e. the point $x^k = 0$ belongs always to ζ). Obviously, there are infinitely many such coordinate systems on Σ_t, which differ from each other by an $O(3)$-rotation. To fix uniquely coordinates (x^k), we choose the unique boost transformation relating the laboratory time axis $\partial/\partial y^0$ with the four-velocity vector $U := u^\mu \frac{\partial}{\partial y^\mu}$. Next, we define the position of the $\partial/\partial x^k$ – axis on Σ_t by transforming the corresponding $\partial/\partial y^k$ – axis of the laboratory frame by the same boost. The final formula relating Minkowskian coordinates (y^μ) with the new parameters (t, x^k) may be easily calculated (see e. g. [3]) from the above definition:

$$
\begin{aligned}
y^0(t, x^l) &:= t + \frac{1}{\sqrt{1 - \mathbf{v}^2(t)}} \, x^l v_l(t) \, , \\
y^k(t, x^l) &:= q^k(t) + \left(\delta^k_l + \varphi(\mathbf{v}^2) v^k v_l \right) x^l \, ,
\end{aligned}
\tag{A.1}
$$

where we denote $\varphi(z) := \frac{1}{z} \left(\frac{1}{\sqrt{1-z}} - 1 \right) = \frac{1}{\sqrt{1-z}(1+\sqrt{1-z})}$.

Observe, that the particle's Coulomb field has in this co-moving frame extremely simple form:

$$
\mathcal{D}_0(\mathbf{x}) = \frac{e\mathbf{x}}{4\pi r^3} \, , \qquad \mathcal{B}_0(\mathbf{x}) = 0 \, ,
\tag{A.2}
$$

where $r := |\mathbf{x}|$. That is why the calculations in this frame are much easier than in the laboratory one.

Let $\mathcal{D}_k$ and $\mathcal{B}_k$ denote the rest-frame components of the electric and magnetic field. They are related to D_k and B_k as follows:

$$
\begin{aligned}
\mathcal{D}_k(\mathbf{x}, t; \mathbf{q}, \mathbf{v}) &= \frac{1}{\sqrt{1 - \mathbf{v}^2}} \left[\left(\delta^l_k - \sqrt{1 - \mathbf{v}^2} \varphi(\mathbf{v}^2) v^l v_k \right) D_l(y) - \epsilon_{kij} v^i B^j(y) \right] , &\text{(A.3)} \\
\mathcal{B}_k(\mathbf{x}, t; \mathbf{q}, \mathbf{v}) &= \frac{1}{\sqrt{1 - \mathbf{v}^2}} \left[\left(\delta^l_k - \sqrt{1 - \mathbf{v}^2} \varphi(\mathbf{v}^2) v^l v_k \right) B_l(y) + \epsilon_{kij} v^i D^j(y) \right] , &\text{(A.4)}
\end{aligned}
$$

(the matrix $(\delta^l_k - \sqrt{1 - \mathbf{v}^2} \, \varphi(\mathbf{v}^2) v^l v_k)$ comes from the boost transformation).

The field evolution with respect to the above *non inertial* frame is a superposition of the following three transformations (cf. [1], [2], [3]):

- time-translation in the direction of U,

- boost in the direction of the particle's acceleration a^k,

- purely spatial $O(3)$-rotation around the vector ω_m,

where

$$
\begin{aligned}
a^k &:= \frac{1}{1 - \mathbf{v}^2} \left(\delta^k_l + \varphi(\mathbf{v}^2) v^k v_l \right) \dot{v}^l \, , &\text{(A.5)} \\
\omega_m &:= \frac{1}{\sqrt{1 - \mathbf{v}^2}} \, \varphi(\mathbf{v}^2) v^k \dot{v}^l \epsilon_{klm} \, . &\text{(A.6)}
\end{aligned}
$$

Therefore, the Maxwell equations read (cf. [2], [3]):

$$\dot{\mathcal{D}}^n = \sqrt{1-\mathbf{v}^2}\,\frac{\partial}{\partial x^m}\left[\left(\epsilon^{mk}{}_i\mathcal{D}^n - \epsilon^{nk}{}_i\mathcal{D}^m\right)\omega_k x^i - \epsilon^{mn}{}_k(1+a^i x_i)\mathcal{B}^k\right]\,, \tag{A.7}$$

$$\dot{\mathcal{B}}^n = \sqrt{1-\mathbf{v}^2}\,\frac{\partial}{\partial x^m}\left[\left(\epsilon^{mk}{}_i\mathcal{B}^n - \epsilon^{nk}{}_i\mathcal{B}^m\right)\omega_k x^i + \epsilon^{mn}{}_k(1+a^i x_i)\mathcal{D}^k\right]\,, \tag{A.8}$$

(the factor $\sqrt{1-\mathbf{v}^2}$ is necessary, because the time t, which we used to parametrize the particle's trajectory, is not a proper time along ζ but the laboratory time).

On the other hand, the time derivative with respect to the co-moving frame may be written as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^k\frac{\partial}{\partial q^k} + \dot{v}^k\frac{\partial}{\partial v^k} = \left(\frac{\partial}{\partial t}\right)_U + \dot{v}^k\frac{\partial}{\partial v^k}\,. \tag{A.9}$$

Therefore, taking into account (A.7) and (A.8) we obtain:

$$\left(\frac{\partial}{\partial t}\right)_U \mathcal{D}^n = \sqrt{1-\mathbf{v}^2}\,\epsilon^{nmk}\partial_m\mathcal{B}_k\,, \tag{A.10}$$

$$\left(\frac{\partial}{\partial t}\right)_U \mathcal{B}^n = -\sqrt{1-\mathbf{v}^2}\,\epsilon^{nmk}\partial_m\mathcal{D}_k\,, \tag{A.11}$$

and

$$\frac{\partial}{\partial v^l}\mathcal{D}^n = \sqrt{1-\mathbf{v}^2}\,\partial_m\left[\frac{\partial\omega_k}{\partial\dot{v}^l}\left(\epsilon^{mk}{}_i\mathcal{D}^n - \epsilon^{nk}{}_i\mathcal{D}^m\right)x^i - \frac{\partial a^k}{\partial\dot{v}^l}\epsilon^{mn}{}_k x_i\mathcal{B}^k\right]\,, \tag{A.12}$$

$$\frac{\partial}{\partial v^l}\mathcal{B}^n = \sqrt{1-\mathbf{v}^2}\,\partial_m\left[\frac{\partial\omega_k}{\partial\dot{v}^l}\left(\epsilon^{mk}{}_i\mathcal{D}^n - \epsilon^{nk}{}_i\mathcal{D}^m\right)x^i + \frac{\partial a^k}{\partial\dot{v}^l}\epsilon^{mn}{}_k x_i\mathcal{B}^k\right]\,. \tag{A.13}$$

To calculate the derivatives of $\mathcal{D}^k$ and $\mathcal{B}^k$ with respect to the particle's position observe, that

$$\frac{\partial}{\partial y^k} = -\frac{v_k}{\sqrt{1-\mathbf{v}^2}}\,U + \left(\delta^i_k + \varphi(\mathbf{v}^2)v^i v_k\right)\frac{\partial}{\partial x^i}\,. \tag{A.14}$$

Therefore

$$\frac{\partial}{\partial q^k}\mathcal{D}^n = -\frac{v_k}{\sqrt{1-\mathbf{v}^2}}\,\epsilon^{nmi}\partial_m\mathcal{B}_i + \left(\delta^i_k + \varphi(\mathbf{v}^2)v^i v_k\right)\partial_i\mathcal{D}^n\,, \tag{A.15}$$

$$\frac{\partial}{\partial q^k}\mathcal{B}^n = \frac{v_k}{\sqrt{1-\mathbf{v}^2}}\,\epsilon^{nmi}\partial_m\mathcal{D}_i + \left(\delta^i_k + \varphi(\mathbf{v}^2)v^i v_k\right)\partial_i\mathcal{B}^n\,. \tag{A.16}$$

Now, using (A.10)–(A.13) and (A.15)–(A.16) we prove Lemmas 1–4.

1. Proof of Lemma 1:

Observe, that "interaction static moment" in the particle's rest frame reads:

$$R^{int}_k := \int_{\Sigma_t} x_k(\mathcal{D}_0\mathcal{D} + \mathcal{B}_0\mathcal{B})\,d^3x = \frac{e}{4\pi}\int_{\Sigma_t}\frac{x_k x^i}{r^3}\mathcal{D}^i\,d^3x\,. \tag{A.17}$$

Taking into account that

$$r_k^{int} = \frac{1}{\sqrt{1-\mathbf{v}^2}} \left(\delta_k^i - \sqrt{1-\mathbf{v}^2}\, \varphi(\mathbf{v}^2) v^i v_k \right) R_i^{int} \tag{A.18}$$

we obtain the formula for π_k in terms of R_i^{int}:

$$\pi_k = -\frac{1}{\sqrt{1-\mathbf{v}^2}} \left(\delta_k^i + \varphi(\mathbf{v}^2) v^i v_k \right) R_i^{int} \;. \tag{A.19}$$

Now, using (A.12) one gets:

$$\frac{\partial}{\partial v^l}\, R_i^{int} = \sqrt{1-\mathbf{v}^2} \left\{ \frac{\partial a^m}{\partial \dot{v}^l} \mathbf{X}_{im} - \frac{\partial \omega^m}{\partial \dot{v}^l} \epsilon_{im}{}^j R_j^{int} \right\} \;, \tag{A.20}$$

where

$$\mathbf{X}_{im} = \frac{e}{4\pi} \epsilon_{ijk} \int_{\Sigma_t} \frac{x^j x_m}{r^3} \mathcal{B}^k \, d^3 x \;. \tag{A.21}$$

Therefore

$$\frac{\partial \pi_k}{\partial v^l} - \frac{\partial \pi_l}{\partial v^k} = A^i{}_{kl} R_i^{int} - B_{kl}^{im} \mathbf{X}_{im} \;, \tag{A.22}$$

where

$$\begin{aligned}
A^i{}_{kl} &= \frac{\partial}{\partial v^k} \left[\frac{1}{\sqrt{1-\mathbf{v}^2}} \left(\delta_l^i + \varphi(\mathbf{v}^2) v^i v_l \right) \right] - \frac{\partial}{\partial v^l} \left[\frac{1}{\sqrt{1-\mathbf{v}^2}} \left(\delta_k^i + \varphi(\mathbf{v}^2) v^i v_k \right) \right] \\
&\quad + \epsilon^i{}_{jm} \left[\left(\delta_k^j + \varphi(\mathbf{v}^2) v^j v_k \right) \frac{\partial \omega^m}{\partial \dot{v}^l} - \left(\delta_l^j + \varphi(\mathbf{v}^2) v^j v_l \right) \frac{\partial \omega^m}{\partial \dot{v}^k} \right] \;, \tag{A.23}
\end{aligned}$$

$$\begin{aligned}
B_{kl}^{im} &= \left(\delta_k^i + \varphi(\mathbf{v}^2) v^i v_k \right) \frac{\partial a^m}{\partial \dot{v}^l} - \left(\delta_l^i + \varphi(\mathbf{v}^2) v^i v_l \right) \frac{\partial a^m}{\partial \dot{v}^k} \\
&= (1-\mathbf{v}^2) \left(\frac{\partial a^i}{\partial \dot{v}^k} \frac{\partial a^m}{\partial \dot{v}^l} - \frac{\partial a^i}{\partial \dot{v}^l} \frac{\partial a^m}{\partial \dot{v}^k} \right) \;. \tag{A.24}
\end{aligned}$$

Using the following properties of the function $\varphi(z)$:

$$\begin{aligned}
2\varphi(z) - (1-z)^{-1} + z\varphi^2(z) &= 0 \;, \tag{A.25} \\
2\varphi'(z) - (1-z)^{-1}\varphi(z) - \varphi^2(z) &= 0 \;, \tag{A.26}
\end{aligned}$$

one easily shows that $A^i{}_{kl} \equiv 0$. Moreover, observe that B_{kl}^{im} defined in (A.24) is antisymmetric in (im). Therefore, to prove (34) it is sufficient to show that the quantity $\mathbf{X}_{im}$ is symmetric in (im). Taking into account that $\mathcal{B}^k = \epsilon^{klm} \partial_l \mathcal{A}_m$, where $\mathcal{A}_m$ stands for the rest-frame components of vector potential, one immediatelly gets:

$$\epsilon_{ijk} \int_{\Sigma_t} \frac{x^j x_m}{r^3} \mathcal{B}^k \, d^3 x = \int_{\Sigma_t} r^{-5} (\mathcal{A}_k x^k)(3 x_i x_m - r^2 g_{im}) \, d^3 x \;, \tag{A.27}$$

which ends the proof of (34).

2. Proof of Lemma 2:

To prove (35) observe that

$$\frac{\partial \pi_k}{\partial t} + v^l \frac{\partial \pi_k}{\partial q^l} = \left(\frac{\partial}{\partial t}\right)_U \pi_k = -\frac{1}{\sqrt{1-\mathbf{v}^2}} \left(\delta_k^i + \varphi(\mathbf{v}^2)v^i v_k\right) \left(\frac{\partial}{\partial t}\right)_U R_i^{int} . \qquad (A.28)$$

Now, using (A.10) we obtain

$$\begin{aligned}
\left(\frac{\partial}{\partial t}\right)_U R_i^{int} &= \sqrt{1-\mathbf{v}^2} \int_{\Sigma_t} \frac{x_i x_k}{r^3} \epsilon^{kjm} \partial_j \mathcal{B}_m \, d^3x \\
&= \sqrt{1-\mathbf{v}^2} \int_{\Sigma_t} \partial_j \left(\frac{x_i x_k}{r^3} \epsilon^{kjm} \mathcal{B}_m\right) d^3x + \sqrt{1-\mathbf{v}^2} \int_{\Sigma_t} \epsilon_i^{km} \frac{x_k}{r^3} \mathcal{B}_m \, d^3x .
\end{aligned} \qquad (A.29)$$

Due to the Gauss theorem

$$\int_{\Sigma_t} \partial_j \left(\frac{x_i x_k}{r^3} \epsilon^{kjm} \mathcal{B}_m\right) d^3x = \int_{\partial \Sigma_t} \frac{x_j x_i x_k}{r^4} \epsilon^{kjm} \mathcal{B}_m \, d\sigma \equiv 0 , \qquad (A.30)$$

where $d\sigma$ denotes the surface measure on $\partial \Sigma_t$. Moreover, observe that "interaction momentum" in the particle's rest-frame reads:

$$P_i^{int} := \epsilon_{ikm} \int_{\Sigma_t} (\mathcal{D}_0^k \mathcal{B}^m + \mathcal{D}^k \mathcal{B}_0^m) \, d^3x = \frac{e}{4\pi} \epsilon_{ikm} \int_{\Sigma_t} \frac{x^k}{r^3} \mathcal{B}^m \, d^3x . \qquad (A.31)$$

Therefore

$$\left(\frac{\partial}{\partial t}\right)_U R_i^{int} = \sqrt{1-\mathbf{v}^2} \, P_i^{int} . \qquad (A.32)$$

Using the relation between p_k^{int} and P_i^{int}

$$p_k^{int} = \left(\delta_k^i + \varphi(\mathbf{v}^2)v^i v_k\right) P_i^{int} \qquad (A.33)$$

we finally get (35).

3. Proof of Lemma 3:

Using (A.13) and (A.15) we obtain:

$$\frac{\partial}{\partial v^l} P_i^{int} = \sqrt{1-\mathbf{v}^2} \left\{ \frac{\partial \omega^m}{\partial \dot{v}^l} \epsilon_{im}{}^j P_j^{int} - \frac{\partial a^m}{\partial \dot{v}^l} \mathbf{Y}_{mi} \right\} , \qquad (A.34)$$

$$\frac{\partial}{\partial v^l} R_i^{int} = \frac{v_l}{\sqrt{1-\mathbf{v}^2}} P_i^{int} + \left(\delta_l^j + \varphi(\mathbf{v}^2)v^j v_l\right) \mathbf{Y}_{ij} , \qquad (A.35)$$

where

$$\mathbf{Y}_{ij} := \frac{e}{4\pi} \int_{\Sigma_t} \frac{x_i}{r^5} (3x^k x_j - r^2 \delta_j^k) \mathcal{D}_k \, d^3x . \qquad (A.36)$$

14

Now, taking into account (A.19) and (A.33) we have

$$\frac{\partial p_k^{int}}{\partial v^l} - \frac{\partial \pi_l}{\partial q^k} = C^i{}_{kl} P_i^{int} \ , \tag{A.37}$$

where

$$\begin{aligned}
C^i{}_{kl} \ &:= \ \frac{\partial}{\partial v^l} \left(\delta_k^i + \varphi(\mathbf{v}^2) v^i v_k \right) - \sqrt{1 - \mathbf{v}^2} \left(\delta_k^j + \varphi(\mathbf{v}^2) v^j v_k \right) \frac{\partial \omega^m}{\partial \dot{v}^l} \ \epsilon_{jm}{}^i \\
&+ \ \frac{v_k}{1 - \mathbf{v}^2} \left(\delta_l^i + \varphi(\mathbf{v}^2) v^i v_l \right) \ .
\end{aligned} \tag{A.38}$$

One easily shows that due to properties (A.25)–(A.26) $C^i{}_{kl} \equiv 0$, which ends the proof of (41).

4. Proof of Lemma 4:

Finally, to prove (42) let us observe that

$$\frac{\partial p_k^{int}}{\partial t} + v^l \frac{\partial p_k^{int}}{\partial q^l} = \left(\frac{\partial}{\partial t} \right)_U p_k^{int} = \left(\delta_k^i + \varphi(\mathbf{v}^2) v^i v_k \right) \left(\frac{\partial}{\partial t} \right)_U P_i^{int} \ . \tag{A.39}$$

Now, due to (A.11) we get

$$\begin{aligned}
\left(\frac{\partial}{\partial t} \right)_U P_i^{int} &= \sqrt{1 - \mathbf{v}^2} \ \frac{e}{4\pi} \int_{\partial \Sigma_t} \frac{1}{r^2} \mathcal{D}_i \, d\sigma \\
&= -\sqrt{1 - \mathbf{v}^2} \ \frac{e}{4\pi} \ \lim_{r_0 \to 0} \int_{S(r_0)} \frac{1}{r^2} \mathcal{D}_i \, d\sigma = -\sqrt{1 - \mathbf{v}^2} \ e \mathcal{D}_i(t, 0) \ , \tag{A.40}
\end{aligned}$$

where we choose as two pieces of a boundary $\partial \Sigma_t$ a sphere at infinity and a sphere $S(r_0)$. Using the fact that in the Heaviside-Lorentz system of units $\mathcal{D}_k = \mathcal{E}_k$ and taking into account the formula (A.3) we finally obtain (42).

References

[1] D. Chruściński, J. Kijowski, J. Geom. Phys. **20** (1996) 393.

[2] J. Kijowski, Gen. Rel. and Grav. **26** (1994) 167,
J. Kijowski, Acta Phys. Polon. **A85** (1994) 771.

[3] J. Kijowski, D. Chruściński, Gen. Rel. and Grav. **27** (1995) 267.

[4] R. Abraham, J. E. Marsden, *Foundation of Mechanics*, Benjamin, New York, 1978,

[5] J. M. Souriau, *Structures des Systemes Dynamiques*, Dunod, Paris 1970,
S. Sternberg, Proc. Nat. Acad. Sci. **74** (1977) 5253,
A. Weinstein, Lett. Math. Phys. **2** (1977) 417.